Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021–8928(99)00079-9
J. Appl. Maths Mechs, Vol. 63, No. 4, pp. 641–649, 1999

0021–8928/99/$—see front matter

THE INDENTATION OF A PUNCH IN THE FORM OF AN ELLIPTIC PARABOLOID INTO THE PLANE BOUNDARY OF AN ELASTIC BODY†

I. I. ARGATOV
St Petersburg

(*Received* 9 *July* 1998)

The three-dimensional contact problem for an elastic body of arbitrary geometry with a single plane face, into which a punch in the shape of an elliptic paraboloid is indented, is considered. The curvilinear boundary of the body is partially clamped, and the remaining boundary (outside the contact region) is stress-free. It is assumed that the dimensions of the contact area are small compared with the characteristic dimension of the body. Using the method of matched asymptotic expansions a model problem of unilateral contact without friction is derived for the boundary layer, which is solved using the apparatus of Hertz's theory. Asymptotic models of the contact interaction of different degrees of accuracy are constructed, including corrections to the geometry and clamping conditions of the elastic body. The sensitivity of the parameters of the elliptic region of the contact to these factors is investigated.

1. FORMULATION OF THE PROBLEM

A punch in the shape of an elliptic paraboloid, acted upon by a system of forces with resultant Q and moments M_1 and M_2 about horizontal axes, is gradually impressed without friction into the plane section Γ_c of the boundary of an elastic body Ω to a depth δ_0; the body is clamped on the part Γ_u, and is stress-free on Γ_σ and Γ_c outside the contact region (see Fig. 1).

Suppose l is the radius of the largest hemisphere contained in Ω with centre at the point O. We will denote by ε a small positive parameter and put

$$R_1 = \varepsilon R_1^*, \quad R_2 = \varepsilon R_2^*; \quad \delta_0 = \varepsilon\delta_0^* \tag{1.1}$$

Here R_1 and R_2 are the radii of curvature of the principal normal sections of the surface of the punch at its vertex, where the quantities δ_0^* and R_1^* and R_2^* are comparable with l.

The vector $\mathbf{u} = (u_1, u_2, u_3)$ of the displacements of points of the elastic body Ω satisfies the problem

$$L(\nabla_x)\mathbf{u}(\varepsilon;\mathbf{x}) \equiv -\mu\nabla_x \cdot \nabla_x \mathbf{u}(\varepsilon;\mathbf{x}) - \frac{\mu}{1-2\nu}\nabla_x\nabla_x \cdot \mathbf{u}(\varepsilon;\mathbf{x}) = 0, \quad \mathbf{x}\in\Omega \tag{1.2}$$

$$\sigma_{31}(\mathbf{u};\mathbf{x}) = \sigma_{32}(\mathbf{u};\mathbf{x}) = 0, \quad \mathbf{x}\in\Gamma_c \tag{1.3}$$

$$\begin{gathered} u_3(\varepsilon;\mathbf{x}) \geqslant \varepsilon\delta_0^* - \frac{x_1^2}{2\varepsilon R_1^*} - \frac{x_2^2}{2\varepsilon R_2^*}, \quad \sigma_{33}(\mathbf{u};\mathbf{x}) \leqslant 0 \\ \left[u_3(\varepsilon;\mathbf{x}) - \varepsilon\delta_0^* + \frac{x_1^2}{2\varepsilon R_1^*} + \frac{x_2^2}{2\varepsilon R_2^*}\right]\sigma_{33}(\mathbf{u};\mathbf{x}) = 0, \quad \mathbf{x}\in\Gamma_c \end{gathered} \tag{1.4}$$

$$\boldsymbol{\sigma}^{(n)}(\mathbf{u};\mathbf{x}) = 0, \quad \mathbf{x}\in\Gamma_\sigma; \quad \mathbf{u}(\varepsilon;\mathbf{x}) = 0, \quad \mathbf{x}\in\Gamma_u \tag{1.5}$$

Here μ is the shear modulus, ν is Poisson's ratio, $\sigma_{3j}(\mathbf{u})$ are the components of the stress tensor and $\boldsymbol{\sigma}^{(n)}$ is the stress vector on an area with normal $\mathbf{n}$.

The contact area is unknown in advance and is determined by the condition for the contact pressures to be positive

†*Prikl. Mat. Mekh.* Vol. 63, No. 4, pp. 671–679, 1999.

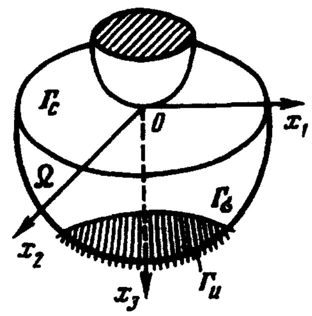

Fig. 1.

$$p(x_1, x_2) = -\sigma_{33}(\mathbf{u}; x_1, x_2, 0)$$

Clearly R_1, R_2 and δ_0 define the dimensions of the contact area. In case (1.1) the latter turn out to be small compared with the characteristic dimension l of the body Ω.

One of the purposes of this paper is to establish the relation between the force Q and the displacement δ_0, and also to determine the moments M_1 and M_2. By Hertz's theory, by virtue of equalities (1.1), we can write

$$Q = \varepsilon^2 Q^* \tag{1.6}$$

Problem (1.2)–(1.5) and related problems, including the unilateral contact conditions (1.4), have been investigated using the theory of variational inequalities ([1–3], etc.). Asymptotic methods of investigating variational inequalities have been developed ([4, 5], etc.). Analytic solutions of the contact problem have been constructed for the case of a layer [6] and a wedge [7] by the "large λ" method [8]. In this paper we use the method of matched asymptotic expansions [9, 10].

2. THE OUTER ASYMPTOTIC EXPANSION

We will denote Green's vector function with a pole at the origin of coordinates by $\mathbf{G}$. This satisfies the relations

$$L(\nabla_x)\mathbf{G}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega \tag{2.1}$$

$$\sigma_{3j}(\mathbf{G}; \mathbf{x}) = 0, \quad j = 1, 2, 3; \quad \mathbf{x} \in \Gamma_c \setminus O \tag{2.2}$$

$$\mathbf{G}(\mathbf{x}) = \mathbf{T}(\mathbf{x}) + O(1), \quad |\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2} \to 0 \tag{2.3}$$

$$\sigma^{(n)}(\mathbf{G}; \mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_\sigma; \quad \mathbf{G}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_u \tag{2.4}$$

Here $\mathbf{T}$ is the solution of the Boussinesq problem (see, for example, [11]) on the action on the boundary of an elastic half-space $x_3 > 0$ of a single point force, directed along the Ox_3 axis

$$\begin{aligned} 4\pi\mu T_i(\mathbf{x}) &= x_i x_3 |\mathbf{x}|^{-3} - (1 - 2\nu)x_i |\mathbf{x}|^{-1} (|\mathbf{x}| + x_3)^{-1}, \quad i = 1,2 \\ 4\pi\mu T_3(\mathbf{x}) &= x_3^2 |\mathbf{x}|^{-3} + 2(1 - \nu)|\mathbf{x}|^{-1} \end{aligned} \tag{2.5}$$

Suppose (to simplify the formulae) that the body Ω is wholly contained in the half-space $x_3 > 0$. Then the regular component of Green's vector function

$$\mathbf{G}(\mathbf{x}) = \mathbf{T}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \tag{2.6}$$

annuls the discrepancy in boundary conditions (2.4), which arise when the sum (2.6) is substituted there, i.e.

$$\sigma^{(n)}(\mathbf{g}; \mathbf{x}) = -\sigma^{(n)}(\mathbf{T}; \mathbf{x}), \quad \mathbf{x} \in \Gamma_\sigma; \quad \mathbf{g}(\mathbf{x}) = -\mathbf{T}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_u \tag{2.7}$$

Moreover, as $|\mathbf{x}| \to 0$ the following expansions hold

$$\mathbf{g}(\mathbf{x}) = \mathbf{g}(0) + \sum_{k=1}^{6} g_{1,k}\mathbf{V}_k^1(\mathbf{x}) + \sum_{k=1}^{9} g_{2,k}\mathbf{V}_k^2(\mathbf{x}) + O(|\mathbf{x}|^3) \tag{2.8}$$

$$\frac{2\pi\mu}{1-\nu} g_3(x_1,x_2,0) = A + B_1x_1 + B_2x_2 + C_{11}x_1^2 + 2C_{12}x_1x_2 + C_{22}x_2^2 + O((x_1^2+x_2^2)^{3/2}) \tag{2.9}$$

Since the vector function $\mathbf{g}$ in the neighbourhood of the origin of coordinates satisfies the homogeneous Lamé system (2.1) and boundary conditions (2.2), its Maclaurin series, generally speaking, will contain $3(m+1)$ homogeneous vector polynomials $\mathbf{V}_k^m$ of degree m ([12, Chapter 13, Section 1] and [13, Section 5.3])

$$\mathbf{V}_k^m(tx_1,tx_2,tx_3) = t^m\mathbf{V}_k^m(\mathbf{x}) \tag{2.10}$$

The constants on the right-hand side of (2.9) are determined by the shape and dimensions of the body Ω and the nature of its clamping, and depend on the value of Poisson's ratio. If L has the dimension of length, the dimensions of the quantities A, B_i and C_{ij} $(i,j = 1, 2)$ are L^{-1}, L^{-2} and L^{-3}, respectively.

When the parameter ε is reduced the area of distribution of the contact pressures contracts to the punch vertex. Hence, at a distance from it the stress–strain state of the body Ω is approximately described by the solution of the problem of the action on its boundary at the point O of a point force of value Q (see (1.6))

$$\mathbf{v}(\varepsilon;\mathbf{x}) = \varepsilon^2 Q^*\mathbf{G}(\mathbf{x}) \tag{2.11}$$

3. THE INNER ASYMPTOTIC EXPANSION

In the region of local perturbations we will introduce the "extended" coordinates

$$\boldsymbol{\xi} = (\xi_1,\xi_2,\xi_3); \quad \xi_i = \varepsilon^{-1}x_i \tag{3.1}$$

Here parts of the body boundary, on which boundary conditions (1.5) are specified, are further than $\varepsilon^{-1}l$ from the origin of coordinates (bearing the new scale in mind). As a result of this the problem for the inner asymptotic expansion is formulated in the half-space $\xi_3 \geqslant 0$. Relations (1.2)–(1.4) give

$$L(\nabla_\xi)\mathbf{w}(\varepsilon;\boldsymbol{\xi}) = 0, \quad \xi_3 < 0 \tag{3.2}$$

$$\sigma_{31}(\mathbf{w};\boldsymbol{\xi}) = \sigma_{32}(\mathbf{w};\boldsymbol{\xi}) = 0, \quad \xi_3 = 0 \tag{3.3}$$

$$\begin{gathered} w_3(\varepsilon;\boldsymbol{\xi}) \geqslant \varepsilon(\delta_0^* - \Phi^*(\xi_1,\xi_2)), \quad \sigma_{33}(\mathbf{w};\boldsymbol{\xi}) \leqslant 0 \\ [w_3(\varepsilon;\boldsymbol{\xi}) - \varepsilon(\delta_0^* - \Phi^*(\xi_1,\xi_2))]\sigma_{33}(\mathbf{w};\boldsymbol{\xi}) = 0, \quad \xi_3 = 0 \\ \Phi^*(\xi_1,\xi_2) = (2R_1^*)^{-1}\xi_1^2 + (2R_2^*)^{-1}\xi_2^2 \end{gathered} \tag{3.4}$$

Formulae (3.2)–(3.4) are closed by the condition of the behaviour of $\mathbf{w}(\varepsilon;\boldsymbol{\xi})$ as $|\boldsymbol{\xi}| \to \infty$, which we obtain by matching with (2.11).

In the matching region $\{\mathbf{x}: \sqrt{(\varepsilon)}l/2 \leqslant |\mathbf{x}| \leqslant \sqrt{(\varepsilon)}l\}$, for small values of ε, we obtain

$$\mathbf{v}(\varepsilon;\varepsilon\boldsymbol{\xi}) = \varepsilon^2 Q^*\left[\varepsilon^{-1}\mathbf{T}(\boldsymbol{\xi}) + \mathbf{g}(0) + \varepsilon\sum_{k=1}^{6} g_{1,k}\mathbf{V}_k^1(\boldsymbol{\xi}) + O(\varepsilon)\right] \tag{3.5}$$

(we have grouped the binomial expansion (2.8) together with relations (2.11) and (2.6), we have made the replacement (3.1) and we have taken into account relations (2.5) and (2.10)). Hence, from expression (3.5) we will have

$$\mathbf{w}(\varepsilon;\boldsymbol{\xi}) = \varepsilon Q^*\mathbf{T}(\boldsymbol{\xi}) + \varepsilon^2\mathbf{V}^*(\varepsilon;\boldsymbol{\xi}) + O(|\boldsymbol{\xi}|^{-2}), \quad |\boldsymbol{\xi}| \to \infty \tag{3.6}$$

$$\mathbf{V}^*(\varepsilon;\boldsymbol{\xi}) = Q^*[\mathbf{g}(0) + \varepsilon \sum_{k=1}^{6} g_{1,k} \mathbf{V}_k^1(\boldsymbol{\xi})] \tag{3.7}$$

Retaining only the first term on the right in (3.6) we arrive at the equations of Hertz's theory. Taking into account the terms from the right-hand side of (3.7), we obtain models which refine it to various degrees of accuracy.

We will represent the solution of problem (3.2)–(3.4), (3.6) in the form

$$\mathbf{w}(\varepsilon;\boldsymbol{\xi}) = \varepsilon^2 \mathbf{V}^*(\varepsilon;\boldsymbol{\xi}) + \varepsilon \mathbf{W}(\varepsilon;\boldsymbol{\xi}) \tag{3.8}$$

The vector $\mathbf{V}^*$ satisfies relations (3.2) and (3.3), where $\sigma_{33}(\mathbf{V}^*; \xi_1, \xi_2, 0) = 0$. The third component of vector function (3.7) when $\xi_3 = 0$ remains the trace (see (2.10))

$$\frac{2\pi\mu}{1-\nu} V_3^*(\varepsilon;\xi_1,\xi_2,0) = Q^*[A + \varepsilon(B_1\xi_1 + B_2\xi_2)] \tag{3.9}$$

Substituting (3.8) into (3.2)–(3.4) and (3.6), we arrive at the problem

$$\begin{aligned}
&L(\nabla_\xi)\mathbf{W}(\varepsilon;\boldsymbol{\xi}) = 0, \quad \xi_3 < 0 \\
&\sigma_{31}(\mathbf{W};\boldsymbol{\xi}) = \sigma_{32}(\mathbf{W};\boldsymbol{\xi}) = 0, \quad \xi_3 = 0 \\
&W_3(\varepsilon;\boldsymbol{\xi}) \geq \delta_0^* - \Phi^*(\xi_1,\xi_2) - \varepsilon V_3^*(\varepsilon;\boldsymbol{\xi}), \quad \sigma_{33}(\mathbf{W};\boldsymbol{\xi}) \leq 0 \\
&[W_3(\varepsilon;\boldsymbol{\xi}) - \delta_0^* + \Phi^*(\xi_1,\xi_2) + \varepsilon V_3^*(\varepsilon;\boldsymbol{\xi})]\sigma_{33}(\mathbf{W};\boldsymbol{\xi}) = 0, \quad \xi_3 = 0 \\
&\mathbf{W}(\varepsilon;\boldsymbol{\xi}) = Q^*\mathbf{T}(\boldsymbol{\xi}) + O(|\boldsymbol{\xi}|^{-2}), \quad |\boldsymbol{\xi}| \to \infty
\end{aligned} \tag{3.10}$$

We obtain its solution by using well-known results (see, for example [14, 15]).

In case (3.9) the contact area is bounded by an ellipsoid.

4. THE FIRST CORRECTION

Suppose $a = \varepsilon a^*$ and e are the major semiaxis and eccentricity of the elliptical contact area. We will assume that the larger of the radii of curvature is denoted by R_1. Then, confining ourselves solely to the first terms in (3.7) and (3.9), we obtain the equations ([15, Chapter 5, Section 6.5])

$$\delta_0^* - \varepsilon\tilde{Q}^* A = \frac{3\tilde{Q}^*}{2a^*}\mathbf{K}(e), \quad \tilde{Q}^* = \frac{(1-\nu)Q^*}{2\pi\mu} \tag{4.1}$$

$$\begin{gathered}
\frac{1}{R_1^*} = \frac{3\tilde{Q}^*}{a^{*3}}\mathbf{D}(e), \quad \frac{1}{R_2^*} = \frac{3\tilde{Q}^*}{a^{*3}}\frac{\mathbf{B}(e)}{1-e^2} \\
\mathbf{D}(e) = e^{-2}[\mathbf{K}(e) - \mathbf{E}(e)], \quad \mathbf{B}(e) = e^{-2}[\mathbf{E}(e) - (1-e^2)\mathbf{K}(e)]
\end{gathered} \tag{4.2}$$

where $\mathbf{K}$ and $\mathbf{E}$ are the complete elliptic integrals of the first and second kind.

Note that the quantity $\tilde{Q}^*$ has dimension L^2.

The eccentricity of the contact area can be found from the equation

$$\frac{R_2^*}{R_1^*} = \frac{(1-e^2)\mathbf{D}(e)}{\mathbf{B}(e)} \tag{4.3}$$

after which we express a^* in terms of $\tilde{Q}^*$ from (4.2) and substitute into (4.1). We obtain

$$\delta_0^* = c_\delta(e)\left(\frac{\tilde{Q}^*}{\sqrt{R}}\right)^{2/3} + \varepsilon\tilde{Q}^* A \tag{4.4}$$

$$R^* = \frac{2R_1^* R_2^*}{R_1^* + R_2^*}, \quad c_\delta(e) = \mathbf{K}(e)\left(\frac{9(1-e^2)}{4\mathbf{E}(e)}\right)^{1/3} \tag{4.5}$$

The equation relating the value of the punch pressing force and the punch displacement can be rewritten, reverting to the use of the real scale, in the form

$$\delta_0 = c_\delta(e)\left(\frac{\tilde{Q}}{\sqrt{R}}\right)^{2/3} + \tilde{Q}A, \quad \tilde{Q} = \frac{(1-\nu)Q}{2\pi\mu} \tag{4.6}$$

where R is the harmonic mean of the radii of curvature R_1 and R_2.

We will now express Q in terms of δ_0. With the same accuracy with which we wrote Eq. (4.4), we put

$$\tilde{Q}^* = \tilde{Q}_0^* + \varepsilon\tilde{Q}_1^*; \quad \tilde{Q}_0^* = \sqrt{R^*}\,\delta_0^{*3/2} c_\delta^{-3/2}(e) \tag{4.7}$$

(the quantity $\tilde{Q}_0^*$ is found from (4.4) with $\varepsilon = 0$).

We substitute (4.7) into (4.4), replace the increment of the first term by its differential and, neglecting quantities of the order of ε^2, we obtain

$$\tilde{Q}_1^* = -\frac{3R^*A}{2c_\delta^3(e)}\delta_0^{*2} \tag{4.8}$$

Thus, (4.6) is supplemented by the following relation

$$\tilde{Q} = \frac{\sqrt{R}\,\delta_0^{3/2}}{c_\delta^{3/2}(e)} - \frac{3RA\delta_0^2}{2c_\delta^3(e)} \tag{4.9}$$

For the characteristic dimension of the contact region (in the "extended" scale) by (4.7) we have the representation

$$a^* = a_0^* + \varepsilon a_1^*; \quad a_0^* = \sqrt{R^*\delta_0^*}\,c_a(e)$$

$$a_1^* = -R^*\delta_0^* A\frac{c_a^2(e)}{3\mathbf{K}(e)}, \quad c_a(e) = \left(\frac{\mathbf{E}(e)}{\mathbf{K}(e)(1-e^2)}\right)^{1/2} \tag{4.10}$$

Finally, the contact pressure can be calculated from the formula ([15, Chapter 5, Section 6.5])

$$p(x_1, x_2) = \frac{3Q}{2\pi a^2\sqrt{1-e^2}}\sqrt{1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{a^2(1-e^2)}} \tag{4.11}$$

5. THE SECOND CORRECTION

Retaining both terms in (3.7) we find that the following equation must be satisfied inside the contact region (see (3.10))

$$W_3(\varepsilon;\xi_1,\xi_2,0) = \delta_0^* - \frac{\xi_1^2}{2R_1^*} - \frac{\xi_2^2}{2R_2^*} - \varepsilon\tilde{Q}^*A - \varepsilon^2\tilde{Q}^*(B_1\xi_1 + B_2\xi_2) =$$

$$= \delta_0^* - \varepsilon\tilde{Q}^*A - \sum_{i=1}^{2}\frac{1}{2R_i^*}(\xi_i - \xi_i^\circ)^2 + O(\varepsilon^4), \quad \xi_i^\circ = -\varepsilon^2\tilde{Q}^*R_i^*B_i$$

(the complete squares are distinguished). Consequently, in the case considered we again arrive at Eqs (4.1)–(4.6) with the sole difference that the centre of the contact spot is shifted to a point with coordinates ξ_i^0 ($i = 1, 2$), or, which is the same thing

$$x_i^\circ = -\frac{(1-\nu)Q}{4\pi\mu} R_i B_i, \quad i = 1,2 \tag{5.1}$$

The law of the contact-pressure distribution has the form (4.11), where on the right we must replace x_1 and x_2 by $x_1 - x^\circ$ and $x_2 - x^\circ$, respectively.

However, formulae (4.7) and (4.8) can be refined to give

$$\tilde{Q}^* = \tilde{Q}_0^* + \varepsilon\tilde{Q}_1^* + \varepsilon^2\tilde{Q}_2^*; \quad \tilde{Q}_2^* = \frac{21R^{*3/2}A^2}{8c_\delta^{9/2}(e)}\delta_0^{*5/2} \tag{5.2}$$

For the moments of the system of loads which keep the punch in a vertical position, we obtain the expressions

$$M_1 = x_2^\circ Q, \quad M_1 = -x_1^\circ Q \tag{5.3}$$

From the representation of the solution of problem (3.10) in the form of the generalized potential of a simple layer, we obtain the expansion

$$\mathbf{W}(\varepsilon;\boldsymbol{\xi}) = Q^*\mathbf{T}(\boldsymbol{\xi}) + \sum_{i=1}^{2} M_i^*\mathbf{S}^{(i)}(\boldsymbol{\xi}) + \sum_{n=0}^{2} M_{2,n}^*\mathbf{S}^{(2,n)}(\boldsymbol{\xi}) + O(|\boldsymbol{\xi}|^{-4}), \quad |\boldsymbol{\xi}|\mapsto\infty$$
$$\mathbf{S}^{(i)}(\boldsymbol{\xi}) = -\frac{\partial\mathbf{T}(\boldsymbol{\xi})}{\partial\xi_2}, \quad \mathbf{S}^{(2)}(\boldsymbol{\xi}) = \frac{\partial\mathbf{T}(\boldsymbol{\xi})}{\partial\xi_1}, \quad \mathbf{S}^{(2,n)}(\boldsymbol{\xi}) = \frac{\partial^2\mathbf{T}(\boldsymbol{\xi})}{\partial\xi_1^{2-n}\partial\xi_2^n} \tag{5.4}$$

The integral characteristics of the pressure possess the following values

$$M_1^* = Q^*\xi_2^\circ, \quad M_2^* = -Q^*\xi_1^\circ \tag{5.5}$$

$$M_{2,0}^* = \frac{Q^*}{2}\left[\frac{1}{5}a^{*2} + (\xi_1^\circ)^2\right], \quad M_{2,1}^* = \xi_1^\circ\xi_2^\circ Q^*, \quad M_{2,2}^* = \frac{Q^*}{2}\left[\frac{1}{5}a^{*2}(1-e^2) + (\xi_2^\circ)^2\right] \tag{5.6}$$

6. REFINEMENT OF THE CONSTRUCTION OF THE ASYMPTOTIC FORM

In (5.4) we make a replacement of the coordinates that is the inverse of (3.1) (the factor ε on the left is set on account of (3.8))

$$\varepsilon\mathbf{W}(\varepsilon;\varepsilon^{-1}\mathbf{x}) \sim \varepsilon^2 Q^*\mathbf{T}(\mathbf{x}) + \varepsilon^3\sum_{i=1}^{2} M_i^*\mathbf{S}^{(i)}(\mathbf{x}) + \varepsilon^4\sum_{n=0}^{2} M_{2,n}^*\mathbf{S}^{(2,n)}(\mathbf{x}) + \ldots \tag{6.1}$$

By the method of matched expansions the terms written on the right in (6.1) mainly define the nature of the singularity in the point O of the outer asymptotic expansion.

When refining (2.11) we bear in mind that coefficients (5.5) are of the order of ε^2 (see (5.1)), and we write the outer asymptotic expansion of the initial problem (1.2)–(1.5) in the form

$$\mathbf{v}(\varepsilon;\mathbf{x}) = Q\mathbf{G}(\mathbf{x}) + \sum_{n=0}^{2} M_{2,n}\mathbf{G}^{(2,n)}(\mathbf{x}) \tag{6.2}$$

where we have introduced the polymoments (as the forces Q are also not defined at this stage)

$$M_{2,n} = \varepsilon^4 M_{2,n}^*, \quad n = 0,1,2 \tag{6.3}$$

The vector function $G^{(2,n)}$ satisfies relations (2.1), (2.2), (2.4) and (2.5) and also the relation

$$\mathbf{G}^{(2,n)}(\mathbf{x}) = \mathbf{S}^{(2,n)}(\mathbf{x}) + O(1), \quad |\mathbf{x}|\mapsto 0$$

The result of matching the outer asymptotic expansion (6.2) and the inner asymptotic expansion, instead of relation (3.6), will be as follows:

$$\mathbf{w}(\varepsilon;\boldsymbol{\xi})=\varepsilon Q^*\mathbf{T}(\boldsymbol{\xi})+\varepsilon\sum_{n=1}^{2} M_{2,n}^*\mathbf{S}^{(2,n)}(\boldsymbol{\xi})+\varepsilon^2\mathbf{V}^*(\varepsilon;\boldsymbol{\xi})+O(|\boldsymbol{\xi}|^{-4}),\quad |\boldsymbol{\xi}|\mapsto\infty \tag{6.4}$$

Here, by (2.8) and (2.9)

$$\mathbf{V}^*(\varepsilon;\boldsymbol{\xi})=Q^*\left[\mathbf{g}(0)+\varepsilon\sum_{k=1}^{6} g_{1,k}\mathbf{V}_k^1(\boldsymbol{\xi})+\varepsilon^2\sum_{k=1}^{9} g_{2,k}\mathbf{V}_k^2(\boldsymbol{\xi})\right]+\varepsilon^2\sum_{n=0}^{2} M_{2,n}^*\mathbf{g}^{(2,n)}(0)$$

$$\frac{2\pi\mu}{1-\nu}V_3^*(\varepsilon;\xi_1,\xi_2,0)=Q^*A+\varepsilon^2\sum_{n=0}^{2} M_{2,n}^*A^{(2,n)}+$$

$$+\ Q^*[\varepsilon(B_1\xi_1+B_2\xi_2)+\varepsilon^2(C_{11}\xi_1^2+2C_{12}\xi_1\xi_2+C_{22}\xi_2^2)] \tag{6.5}$$

By separating the polynomial part, we can represent the solution of the problem for the boundary layer (3.2)–(3.4) and (6.4) in the form (3.8), here the vector **W** satisfies problem (3.10), in which we must make the substitution (6.5) and replace the last relation as follows:

$$\mathbf{W}(\varepsilon;\boldsymbol{\xi})=Q^*\mathbf{T}(\boldsymbol{\xi})+\sum_{n=0}^{2} M_{2,n}^*\mathbf{S}^{(2,n)}(\boldsymbol{\xi})+O(|\boldsymbol{\xi}|^{-4}),\quad |\boldsymbol{\xi}|\mapsto\infty$$

7. THE THIRD CORRECTION

In the general situation the coefficient C_{12} in (6.5) is non-zero. The elliptic contact region then turns out to be turned with respect to the axes of coordinates by a certain angle φ. If $R_1^*=R_2^*$, then φ is defined by the quadratic form

$$\sum_{i,j=1}^{2} C_{ij}\xi_i\xi_j\,(C_{21}=C_{12})$$

If $R_1^*>R_2^*$, then, for small ε, retaining only the leading term we have

$$\varphi=-\varepsilon^3\tilde{Q}_0^*\frac{2R_1^*R_2^*}{R_1^*-R_2^*}C_{12} \tag{7.1}$$

where $\tilde{Q}_0^*$ is given by the second formula of (4.7).

To determine the characteristics of the contact region and the relation between the force and the displacement (retaining only terms up to the order of ε^3 inclusive) we have the equations

$$\delta_0^*-\varepsilon\tilde{Q}^*A-\varepsilon^3\sum_{n=0}^{2}\tilde{M}_{2,n}^{*0}A^{(2,n)}=\frac{3\tilde{Q}^*}{2a^*}\mathbf{K}(e) \tag{7.2}$$

$$\frac{1}{R_1^*}+\varepsilon^3 2\tilde{Q}_0^*C_{11}=\frac{3\tilde{Q}^*}{a^{*3}}\mathbf{D}(e),\quad \frac{1}{R_2^*}+\varepsilon^3 2\tilde{Q}_0^*C_{22}=\frac{3\tilde{Q}^*}{a^{*3}}\frac{\mathbf{B}(e)}{1-e^2} \tag{7.3}$$

The quantities $\tilde{M}_{2,n}^{*0}=(1-\nu)(2\pi\mu)^{-1}\tilde{M}_{2,n}^{*0}$, by (5.6), take the values

$$\tilde{M}_{2,n}^{*0}=\frac{\tilde{Q}_0^*}{10}a_0^{*2},\quad \tilde{M}_{2,1}^{*0}=0,\quad \tilde{M}_{2,2}^{*0}=\frac{\tilde{Q}_0^*}{10}a_0^{*2}(1-e_0^2) \tag{7.4}$$

Here $\tilde{Q}_0^*$, a_0^* and e_0 are the solution of system (7.2), (7.3) when $\varepsilon=0$.

It is assumed that the left-hand side of the first equation of (7.3) is less than the left-hand side of the second, so that the major axis of the ellipse, which bounds the contact region, is oriented along the abscissa axis. When $R_1^*=R_2^*$, the coefficients C_{11} and C_{22} in (7.3) must be replaced by the smaller and larger eigenvalues of the matrix $\|C_{ij}\|$, respectively.

Without loss of accuracy we can derive from (7.3)

$$\frac{R_2^*}{R_1^*}\left(1+\varepsilon^3 2\tilde{Q}_0^*[C_{11}R_1^*-C_{22}R_2^*]\right)=\frac{(1-e^2)\mathbf{D}(e)}{\mathbf{B}(e)} \tag{7.5}$$

Of course, the eccentricity of the contact area, generally speaking, also depends on the force acting on the punch.

We will denote the root of Eq. (4.3) by e_0. Then the result of the approximate solution of Eq. (7.5) has the form

$$e^2 = e_0^2 - \varepsilon^3 2\tilde{Q}_0^*(C_{11}R_1^* - C_{22}R_2^*)\mathbf{c}_e(e_0) \tag{7.6}$$

$$\mathbf{c}_e(e_0) = \frac{2\mathbf{B}(e_0)^2}{\mathbf{E}(e_0)\mathbf{C}(e_0) + \mathbf{B}(e_0)\mathbf{D}(e_0)}; \quad \mathbf{C}(e) = e^{-2}[\mathbf{D}(e) - \mathbf{B}(e)] \tag{7.7}$$

In deriving (7.6) and (7.7) we used formulae for the complete elliptic integrals and their derivatives given in [16, Chapter 9, Section C, Subsection 2.3].

The equation relating the force and the displacement can be obtained by substituting the value of a^* in terms of $\tilde{Q}^*$ and e, found from (7.3), into (7.2). The asymptotic form of its solution (the dependence of $\tilde{Q}^*$ on δ_0^*, up to terms of the order of ε^4) is constructed by using (7.6).

8. REMARKS

At the next stage of refining the construction of the asymptotic form, generally speaking, the conditions which ensure the ellipticity of the contact area will not be satisfied in the model problem for the boundary layer [17, Section 3].

Formulae (4.10), (5.1), (7.1) and (7.6) show the sensitivity of the parameters of the contact spot to the dimensions, shape and clamping conditions of the elastic body.

We recall that $A = 4\pi\mu(1-\nu)^{-1}g_3(0)$, where g_3 is the component of the regular part of Green's vector function normal to the boundary. Using Betti's formula we obtain the representation

$$g_3(0) = -\int_{\Gamma_u} \boldsymbol{\sigma}^{(n)}(\mathbf{G};\mathbf{x})\cdot\mathbf{g}(\mathbf{x})ds + \int_{\Gamma_\sigma} \boldsymbol{\sigma}^{(n)}(\mathbf{g};\mathbf{x})\cdot\mathbf{G}(\mathbf{x})ds$$

When Γ_σ is not present and Γ_u divides the half-space into two parts Ω and Ω_∞, it is easy to prove that the coefficient A is negative. Thus, by (2.7)

$$g_3(0) = -\int_{\Gamma_u} \boldsymbol{\sigma}^{(n)}(\mathbf{g};\mathbf{x})\cdot\mathbf{g}(\mathbf{x})ds + \int_{\Gamma_u} \boldsymbol{\sigma}^{(n)}(\mathbf{T};\mathbf{x})\cdot\mathbf{T}(\mathbf{x})ds$$

Suppose $E(\Omega; \mathbf{u})$ is the potential energy of elastic deformation, corresponding to the displacement field $\mathbf{u}$, stored by the body Γ. Then, using a well-known method [18] and Clapeyron's theorem, we obtain

$$g_3(0) = -2E(\Omega;\mathbf{g}) - 2E(\Omega_\infty;\mathbf{T})$$

I wish to thank S. A. Nazarov for useful discussions.

This research was supported financially by the International Association for Promoting Cooperation with Scientists from the Independent States of the Former Soviet Union (INTAS-96-0876).

REFERENCES

1. FICHERA, G., Boundary-value problems of elasticity with unilateral constraints. In *Handbuch der Physik* (Edited by S. Flügge), B.6a/2, Springer, Berlin, 1972.
2. DUVAUT, G. and LIONS, J.-L., *Inequalities in Mechanics and Physics*. Springer, Berlin, 1976.
3. KRAVCHUK, A. S., The solution of contact problems with known Green's function. *Prikl. Mat. Mekh.*, 1982, **46**, 283–288.
4. NAZAROV, S. A., Perturbations of the solutions of the Signorini problem for a second-order scalar equation. *Mat. Zametki*, 1990, 47, 115–126.
5. ARGATOV, I. I. and NAZAROV, S. A., The asymptotic solution of the problem of an elastic body lying on several small supports. *Prikl. Mat. Mekh.*, 1994, **58**, 110–118.
6. POROSHIN, V. S., The problem of the boundedness of the contact pressures on the contour of a punch elliptic in plane, which interacts with an elastic layer. *Prikl. Mat. Mekh.*, 1984, **48**, 466–472.
7. LUBYAGIN, I. A., POZHARSKII, D. A. and CHEBAKOV, M. I., The indentation of a punch in the form of an elliptic paraboloid into an elastic three-dimensional wedge. *Prikl. Mat. Mekh.*, 1992, **56**, 286–295.
8. ALEKSANDROV, V. M., Asymptotic methods in contact problems of the theory of elasticity. *Prikl. Mat. Mekh.*, 1968, **32**, 4, 672–683.
9. VAN DYKE, M. D., *Perturbation Methods in Fluid Mechanics*. Parabolic Press, Stanford, CA, 1975.
10. IL'IN, A. M., *Matching of Asymptotic Expansions of the Solutions of Boundary-value Problems*. Nauka, Moscow, 1989.
11. NOWACKI, W., *Thermoelasticity*. Addison-Wesley, Reading, MA, 1962.

12. MIKHLIN, S. G., *Linear Partial Differential Equations*. Vyssh. Shkola, Moscow, 1977.
13. ARUTYUNYAN, N. Kh., MOVCHAN, A. B. and NAZAROV, S. A., The behaviour of the solutions of problems of the theory of elasticity in unbounded regions with paraboloidal and cylindrical inclusions or cavities. *Uspekhi Mekh.*, 1987, **10**, 4, 3–91.
14. SHTAYERMAN, I. Ya., *The Contact Problem of the Theory of Elasticity*. Gostekhteoretizdat, Moscow, 1949.
15. LUR'YE, A. I., *Theory of Elasticity*. Nauka, Moscow, 1970.
16. JAHNKE, E. EMDE, F. and LÖSCH, F., *Tables of Higher Functions*. McGraw-Hill, New York, 1960.
17. DOVNOROVICH, V. I., *Three-dimensional Contact Problems of the Theory of Elasticity*. Izd. Belorus. Univ., Minsk, 1959.
18. POLYA, G. and SZEGÖ, G., *Isoperimetric Inequalities in Mathematical Physics*. Princeton University Press, Princeton, NJ, 1951.

Translated by R.C.G.